Implementation of Quaternions and Kepler's Equation in calculating the exact position of a Celestial Body in the Solar System

Aramazaya 13523082
Program Studi Teknik Informatika
Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
[1]13523082@std.stei.itb.ac.id, [2]aramazaya21@gmail.com

***Abstract*— The calculation of celestial bodies has been a tremendous help in the field of astronomy and space exploration. The application of the knowledge of where a celestial body will be in a certain time frame helps in the calculation of Spacecraft Trajectory planning, Navigation system and much more. However, using classical methods such as rotational matrices to calculate the position and rotation of celestial bodies is a herculean task to say the least. The large computational power needed and other limitation such as the Gimbal Lock is too tormenting for real-time simulation. Therefore, the use of quaternions is more encouraged for its scalability, efficiency, and ease of use. In this study, we offer a quaternion-based framework which provides a robust presentation of the three-dimensional rotations. The generalized framework is derived for axial rotation and orbital revolution while including parameters such as tilt and rotation rates for use in all celestial bodies in the solar system. Using this approach is more computationally efficient in handling dynamic updates for real-time simulations. The application of this approach includes, but is not limited to, mission planning, educational tools, and much more. This work provides a better tool for making headways in the field of astrophysics and space exploration.**

***Keywords*—Celestial mechanics, quaternion algebra, orbital dynamics, planetary rotation.**

I. INTRODUCTION

Celestial objects have been observed for thousands of years by humankind. The wonderment of something much larger than ourselves floating in the sky has frightened mankind since the beginning of civilization. Early man believes astronomical object to be deities and gods, while the earliest written records for space observation were produced by the Babylonians (1600 B.C.) who recorded the positions of planets, times of eclipse, etc. It then continues to ancient Greek with Eratosthenes calculating the circumference of the Earth and Heraclides forming the first model of the Geocentric solar system. The Geocentric model was then accepted as doctrine by the Roman Catholic Church and was never contested until the Renaissance. Enter Copernicus, the man who reinvented the heliocentric theory and challenged church doctrine [1]. Space observation then continued to this day with observations from big agencies such as NASA exploring the observable universe for a better understanding of our universe.

In today's time, we use the modelling of celestial bodies in our solar system for many purposes. Satellite deployment needs to follow strict calculation so that it is placed in the proper orbit around celestial bodies. Space mission also follows strict calculation for its landing and trajectory (e.g. Mars rover landing requires a precise model of Mars's rotation and orbit). This knowledge is also used in predicting celestial events (i.e. eclipses), collision awareness, and many education purposes.

Unfortunately, earlier methods of calculation require a significant amount of computational power and complex calculation for varying rotational and orbital properties. Using earlier methods such as the Euler Angles requires complex geometrical calculation to provide a 3x3 transformation matrix which in turn, increases the computational power by leaps and bounds.

In this study, the author aims to use quaternion algebra to provide a framework for simulating the rotation and orbit of all celestial bodies in the Solar System. With a quaternion-based framework, the computational power used to calculate rotation is much less than that of earlier methods such as Euler Angles. Using a combination of Kepler's Law and quaternions, we can calculate the precise location of any celestial bodies in the Solar System using a couple of parameters in a 3D heliocentric frame.

II. THEORETICAL FOUNDATION

A. Orbital Mechanics

Through his analysis of the motions of the planets, Kepler developed a series of principles. These principles is now known as *Kepler's Law of Planetary Motion*. Using Tycho Brahe's observation, Kepler found that the orbit of mars is, although not noticeable to the naked eye, not a circle, but an ellipsoid. The small difference between the circle and the ellipsoid is critical for understanding planetary motion. Kepler generalized the result in to *Kepler's first law of planetary motion*:

"the orbit of all the planets are ellipses"

Kepler's second law deals with the speed with which each planet moves along its ellipses, also known as orbital speed. With Brage's data, Kepler found that the planet speeds up as it comes closer to the Sun and slows down as it pulls away.

Kepler found that it took equal amount of time (t) for Mars to swept the area A and the area B; that is, the area of the region B from 1 to 2 is equal to region A from 3 to 4.

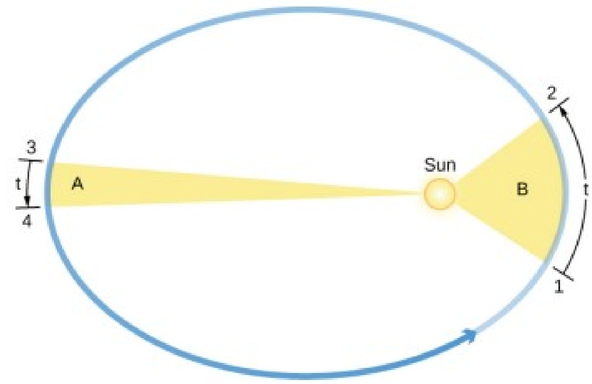


FIGURE I
DEMONSTRATION OF KEPLER'S SECOND LAW OF PLANETARY MOTION

For many years after founding the first two law, Kepler worked to discover mathematical pattern governing the movements of the planets – a "harmony of the sphere" as he called it. In 1619. Kepler discovered a basic relationship between the planets' orbit and their relative distance from the Sun. We define a planet's orbital period, (*P*), as the time it takes a planet to travel once around the sun. The relationship now known as *Kepler's Third Law*, says that a planet's orbital period squared is proportional to the semimajor axis of its orbit cubed, or

$$P^2 = a^3 \tag{2}$$

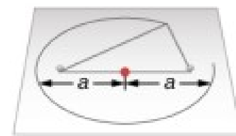
when *P* is measured in years and *a* is measured in a quantity known as *astronomical unit (AU)* which is the average distance between Earth and the Sun, approximately 1.5×10^8 Kilometers. This law applies to all objects orbiting the Sun.

Kepler's three laws of planetary motion can be summarized as follows:

- **Kepler's first law:** Each planet moves around the sun in an orbit that is an ellipse, with the sun at one focus of the ellipse
- **Kepler's second law:** The straight line joining a planet and the Sun sweeps out equal areas in space in equal interval of time.
- **Kepler's third law:** The square of a planet's orbital period is directly proportional to the cube of the semimajor axis of its orbit.

The orbital elements needed in calculating precise polar coordinate with the Sun as the center of reference are the orbit's semi-major axis, eccentricity, inclination, longitude of ascending node, argument of periapsis, and the true anomaly.

The semi-major axis (a) is half the distance of the farthest point from each other in an ellipse (Otherwise known as the major axis). As seen in fig. 2, the semi-major axis is denoted by a and the major axis as 2a.



(b)

FIGURE II
VISUALIZATION OF THE SEMI-MAJOR AXIS

The eccentricity (*e*) of an orbit is the ratio of the distance between the focus and the major axis. The eccentricity denotes how an ellipse would take shape with the lesser it is, the more the ellipse will look like a circle with eccentricity of 0 forming a perfect circle [2].

The inclination (*i*) of an orbit is the angle between the K unit vector and the momentum vector h. This means that the inclination is the tilt of the orbit with the ecliptic plane.

The longitude of ascending node (Ω) is the angle between the I unit vector in the fundamental plane and the point where the satellite cross through the it in a northerly direction.

The argument of periapsis (ω) is the angle between the ascending node and the periapsis point.

While the true anomaly (*v*) is the position of the celestial body in its orbit.

These classical orbital elements is used to denote the orbit of a celestial body and also calculate its precise location in a heliocentric reference frame.

Kepler's equation is used to determine the relationship between time and angular displacement within the orbit. From Kepler's second law, we can derive

$$\sqrt{\frac{a^3}{\mu}} = \frac{(t-T)}{E - e\sin(E)} \tag{2}$$

We can then use the notation for *mean anomaly*, M, as

$$M_0 = E - e\sin(E) = \sqrt{\frac{\mu}{a^3}}(t - T) \tag{3}$$

The mean anomaly at time *t* is given by

$$M(t) = M_0 + n \cdot (t - t_0) \tag{4}$$

From (3), we can find a solution using a couple of methods. The first method is to use the Newton-Raphson Method.

In the Newton-Raphson Method, We first make an initial guess using eccentricity as a basis of our guess.

- **e < 0.3:** $E_0 \approx M$
- **e ≥0.3:** approximate $E_0 = M + e\sin(M)$

Afterwards, we iteratively find E until the Difference is smaller than the tollerance [7].

$$E_{n+1} = E_n - \frac{E_n - e\sin(E_n) - M}{1 - e\cos(E_n)} \quad (5)$$

$$|E_{n+1} - E_n| < \epsilon, \epsilon - Max\ Acceptable\ Error \quad (6)$$

B. Rotational Dynamics

Rotational dynamics focuses on how celestial bodies rotate in the Solar System while following its orbit. For planets, understanding rotation is useful for figuring out seasons, day-night cycle, and much more. Some key concepts which this paper use is:

- **Axial Tilt (θ):** The angle between a planet's orbit and rotation axis. For example, Earth has an axial tilt of 23.5º which is responsible for seasonal variation. An axial tilt can be represented mathematically as a unit vector [3]

$$u = (\sin\theta, 0, \cos\theta) \quad (7)$$

- **Angular Velocity(ω):** The rate of rotation of an object around its axis. Represented as ω with

$$\omega = \frac{2\pi}{T_{rot}} \quad (8)$$

- **Rotational Kinematics:** A point in a rotating body move in circular motion. The velocity at any point is

$$v = \omega \times r \quad (9)$$

With r being the position vector relative to the axis [4].

- **Precession:** Conical motion of the rotation axis caused by external torques.
- **Nutation:** a variation of the inclination of the axis of a rotating object caused by short-term pertrubation [3].

C. Quaternions

In 1843, Sir William Rowan Hamilton, while taking a walk with his wife, made a stunning discovery for a problem he has been interested in. He managed to expand the complex number system from R^2 space to R^3 space. His discovery consisted of a way to multiply unit vectors with each other. From then on, the world is graced by the knowledge of quaternions, a useful framework that we will use in this paper.

Quaternions is a combination of a scalar value and a vector. It takes the form

$$q = a + v = a + bi + cj + dk = (a, v) \quad (10)$$

where a is the scalar part of the equation and v is the vector part. In quaternions, the result of multiplication of unit vectors follows a certain rule where

$$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j \quad (11)$$

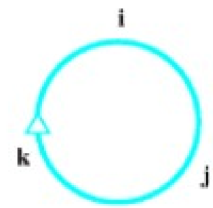


FIGURE III
UNIT VECTOR MULTIPLICATION RULE VISUAL REPRESENTATION

As seen in Figure 3 and (11), if the multiplication follow along the path in figure 3, the result will be positive while if it fights the path, the result will be negative.

Quaternion operation consists of addition, subtraction, and multiplication. The operations follow the same rule as normal algebra operation with the added rule of the unit vector multiplication.

The magnitude of a quaternion is represented by

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2} \quad (12)$$

where a, b, c, d is taken from (10).

The conjugate of a quaternion (10) is represented by

$$\bar{q} = a - v = a - bi - cj - dk \quad (13)$$

while the inverse of a quaternion of the form (10) follows [5].

$$q^{-1} = \frac{1}{q} = \frac{\bar{q}}{\|q\|^2} \quad (14)$$

Using quaternions, one can rotate a vector against an axis.

suppose *p* is a vector in the R^3 space. The image of *p* if *p* is rotated θ degree counter clockwise around the axis *u* will follow

$$p' = qpq^{-1} \quad (15)$$

where

$$p = 0 + xi + yj + zk$$

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{u} \quad (16)$$

and $\hat{u}$ is the unit vector of the axis [6].

III. ANALYSIS

A. System Overview

The framework combines orbital motion and rotational dynamics to model the position and orientation of a celestial body over time in a heliocentric reference frame.

The workflow consists of first inputting the orbital elements of the celestial body. It will then calculate the position of the celestial body in a heliocentric reference frame based on the orbital elements. Next, the framework will calculate the orientation of the body using quaternions as a means of efficient calculation. Lastly, the framework will output the position of the body and the quaternion describing its orientation.

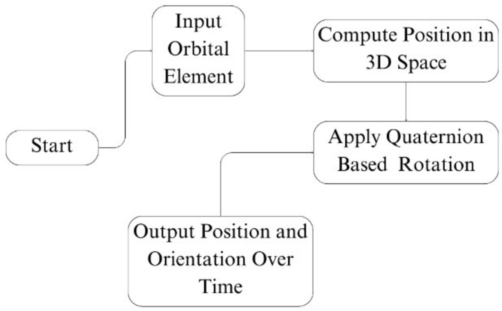


FIGURE IV
WORKFLOW DIAGRAM OF THE FRAMEWORK

B. Parameter Input

For each celestial bodies, input the following parameters:

- **Semi-Major Axis (a):** Used to define the size of the orbit
- **Eccentricity (e):** A value $0 \leq e \leq 1$ Used to define the concavity of the orbit with 0 forming a perfect circle**.**
- **Inclination (i):** the tilt of the orbit with the ecliptic frame.
- **Longitude of Ascending Node (Ω):** Used to denote the orientation of the orbital plane.
- **Argument of Periapsis (ω):** Used to denote the orientation of the closest approach
- **Mean Anomaly (M_0):** the position of the body in it's orbit at a given time to be used as a basis.
- **Axial Tilt (θ):** the angle between the rotation axis and the orbital plane normal [3].
- **Rotation Period (T_{rot}):** Time taken for the body to do one full rotation.
- **Initial Orientation (q_{old}):** Used as a basis for renewing the orientation quaternion. Usually uses (1,0,0,0).

C. Orbital Motion

The first thing to calculate in this module is the Mean Anomaly as it will be used in the next section. We first use (4) to calculate the Mean anomaly at a given time *t*. It will then be used to calculate the Eccentric Anomaly (*E*). Kepler's equation is solved iteratively using the Newton-Raphson method.

The initial step in the Newton-Raphson method is to take an initial guess. If the eccentricity of the orbit is less than 3, it is advisable to use E ≈ M, while if the eccentricity is more than or equal to 3, it is better to use a better approximation

$$E_0 = M + e\sin(M) \tag{17}$$

Next we iteratively calculate E_{n+1} as

$$E_{n+1} = E_n - \frac{E_n - e\sin(E_n) - M}{1 - e\cos(E_n)} \tag{18}$$

until the change in E is below the tolerance (maximum acceptable error [7]).

$$|E_{n-1} - E_n| < \epsilon, \epsilon = Max\ Acceptable\ Error \tag{19}$$

We then use the *E* after convergence for calculating the true anomaly (v) using equation

$$\tan\left(\frac{v}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) \tag{20}$$

and use the true anomaly to calculate the orbital radius for use in converting the location to the cartesian system

$$r = \frac{a(1-e^2)}{1+e\cos(v)} \tag{21}$$

$$x = r\cos(v),\ \ y = r\sin(v) \tag{22}$$

Afterwards, we create a transformation matrix to rotate the cartesian coordinate to the heliocentric 3D frame.

First we use the inclination and the argument of periapsis to find the plane rotation matrix

$$R_{plane} = R_x(i) \cdot R_z(\omega)$$

$$R = R_z(\Omega) \cdot R_{plane} \tag{23}$$

The resulting rotation matrix is

$$\begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix} \tag{24}$$

Lastly, we calculate the heliocentric point by multiplying the rotation matrix with the initial point (20).

D. Rotation Dynamic

The next module in the framework follows is the rotation dynamic module. In it, we calculate the new orientation, taking the form of a quaternion, at time *t.*

Firstly, calculate the angle the body rotated by using angular velocity from (6)

$$\theta = \omega t = \frac{2\pi}{T_{rot}}t \tag{25}$$

Afterward, we use (14) to calculate the quaternion at a time t

$$q(t) = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(u_x i + u_y i + u_z k) \tag{26}$$

where,

$$u = (u_x i + u_y i + u_z k) = (\sin\theta, 0, \cos\theta) \tag{27}$$

The body new orientation is represented as

$$q_{new} = q_{old} \cdot q(t) \tag{28}$$

E. Output Parameter

The framework will output the position from the orbital motion section of the framework and its orientation from the rotation dynamic section.

- **Position:** (x,y,z) coordinate with the Sun as the center.
- **Orientation:** quaternion q(t) representing the body's orientation.

IV. IMPLEMENTATION

A. Overview

The implementation of the framework is divided into 2 components, the orbital motion part and the rotation dynamic part. We use the Newton-Raphson method to calculate the Eccentric anomaly and then calculate the true anomaly and lastly the 3D heliocentric position. The calculation of the orientation uses quaternions and is made very efficient rather than using euler angle, etc. The implementation of the framework is done in python while utilizing libraries like NumPy and SciPy. NumPy is used for ease of use with the matrix operations, quaternion operation, and other array concerned operation, while the SciPy library is used for calculating (18).

B. Code Structure

The code is structured into three parts. The main simulation, the orbital motion, and the rotational dynamic.

C. Orbital Motion

The orbital motion part of the code is called by calling the function orbital_position(). It first run the kepler_solver function which has the algorithm

```
function kepler_solver(M, e)
   E ← M
   while | E – e*sin(E) – M| > tolerance, do
      E ←  E – (E – e*sin(E)-M) / (1-e*cos(E))
   →E
```

it then calculates the true anomaly and orbital radius to find the position in the orbital plane

```
function calculate_true_anomaly(E, e)
   x ← sqrt(1+e)*sin(E/2)
   y ← sqrt(1-e)*cos(E/2)
   v ← 2*atan(x/y)
   → v

function calculate_radius(a, e, v)
   →a*(1-e**2)/(1+e*cos(v))
```

the calculate_true_anomaly function first made the numerator and then the denominator for the final arctan calculation. While the calculate_radius function just straight returns the radius in one single line. Afterward, the code calculate the rotation for the final position coordinate.

```
function make_rotation_matrix(Omega, omega, i)
   R_Z_O ← [[cos(Omega), -sin(Omega), 0] ,
   [sin(Omega), cos(Omega), 0],
   [0, 0, 1]]
   R_Z_o ← [[cos(omega), -sin(omega), 0],
   [sin(omega), cos(omega), 0],
   [0, 0, 1]]
   R_X_i ← [[1,0,0],
   [0, cos(i), -sin(i)],
   [0, sin(i), cos(i)]]
   R ← R_Z_O * R_Z_o * R_X_i
   →R

function calculate_heliocentric(R, r, v)
   →R*[r*cos(v), r*sin(v), 0]
```

The make_rotation_matrix function multiplies all rotation matrix needed for the longitude of ascending node, argument of periapsis, and inclination while the calculate_heliocentric(R, r, v) function returns an array of the modified cartesian coordinates. All the functions was then called within a bigger wrapper function which returns the final solution, which is the calculate_heliocentric function return value.

```
function get_position(a, M, e, Omega, omega, i)
   E ←kepler_solver(M,e)
   v ←calculate_true_anomaly(E,e)
   r ←calculate_radius(a, e,v)
   R ←make_rotation_matrix(Omega, omega, i)
   →calculate_heliocentric(R,r,v)
```

D. Rotation dynamics

In the rotation dynamic part of the code, the program runs a series of function that will return the new quaternion for the orientation of the body.

Firstly, it counted the angle of rotation the body went through in the timeframe t using angular velocity formula. It then input the result into a function that generates a rotated quaternion from the unit vector of the axis and the angle it went through.

```
function quat_from_axis_angle(axis, angle)
   w ← cos(angle/2)
   x ← axis[0]*sin(angle/2)
   y ←axis[1]*sin(angle/2)
   z ←axis[2]*sin(angle/2)
   →[w,x,y,z]
```

The function follows (24) to calculate the scalar value of the quaternion and then the vector value of the quaternion.

E. Main Simulation

The main simulation is a wrapper for all of the other functions in the program. In the main simulation, the program ask for the required input of orbital elements from the user and then calculates the necessary output using the functions in the other parts.

V. CONCLUSION

The ability to locate celestial objects in the Solar System is very important in the pursuit of the unknown. With the knowledge of where a celestial body will be in a specific time, we, humankind, have been able to discover much new knowledge of the universe. Unfortunately, the usage of traditional methods of calculation is widely inefficient in both computing time and power. Therefore, the use of quaternions has been proven to be useful for cutting down computing time and resources. We also managed to build a framework in which anyone can calculate the exact location and orientation of a celestial body in our universe at any given time.

REFERENCES

[1] J. Schombe, "History of Astronomy", Astronomy 121, University of Oregon, Lecture Notes, Accessed: Jan.1, 2025.[Online].Available:https://pages.uoregon.edu/jschombe/ast121/lectures/lec02.html

[2] A. Fraknoi, D. Morrison, S. Wolff, *Astronomy 2e*, XanEdu, 2022.

[3] D. A. Vallado, *Fundamentals of Astrodynamics and Application*s, *4thed*., Microcosm Press, Hawthorne, CA, 2013.

[4] R. Resnick, D. Halliday, *Fundamental of Physics, 9thed*., Wiley and Sons, New York, NY, 2014.

[5] R. Munir, "Aljabar Quaternion (bagian 1)", Aljabar Linear dan Geometri, Institut Teknologi Bandung, Bandung, Jawa Barat, Accessed: Jan. 2, 2025. [Online]. Available: https://informatika.stei.itb.ac.id/~rinaldi.munir/AljabarGeometri/2023-2024/Algeo-25-Aljabar-Quaternion-Bagian1-2023.pdf

[6] R. Munir, "Aljabar Quaternion (bagian 2)", Aljabar Linear dan Geometri, Institut Teknologi Bandung, Bandung, Jawa Barat, Accessed: Jan. 2, 2025. [Online]. Available: https://informatika.stei.itb.ac.id/~rinaldi.munir/AljabarGeometri/2023-2024/Algeo-26-Aljabar-Quaternion-Bagian2-2023.pdf

[7] R. L. Burdenand, J. D. Faires, *Numerical Analysis, 9thed.,* CengageLearning, Boston, MA, 2011

PERNYATAAN

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

Bandung, 27 Desember 2024

Aramazaya 13523082